Lean Modulus

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Chapter 1

Common

This chapter collects definitions and results that are not specific to any one paper: the multigraph infrastructure, and the general theory of families of objects, Fulkerson duality, and modulus. Later papers' chapters specialize this material (e.g. to spanning trees) and build their results on top of it.

When the Lean formalization needs extra explanation, this is indicated inline with an *implementation note* pointing to the relevant file under `docs/` in the GitHub repository, which explains the formalization decisions in more detail.

1.1 Multigraphs

Definition 1 (Multigraph)**.** Following the paragraph preceding Section 1.1 of Albin, Clemens, Hoare, Poggi-Corradini, Sit, and Tymochko (2021): throughout, $G = (V, E)$ denotes a finite, connected multigraph with vertex set V and edge set E. Parallel edges are allowed; self-loops are not.

To formalize this, a multigraph on vertex type V and edge type E consists of a map endpoints: $E \to \mathrm{Sym}^2(V)$ assigning to each edge its (unordered) pair of endpoints. By making the edges a separate type (rather than a relation on V or a subset of unordered pairs of vertices), we implicilty allow parallel edges. They are simply distinct elements of E with the same image under endpoints. Finiteness, connectedness, and the absence of self-loops are not built into the type itself. Instead, they are added explicitly wherever a paper's results need them, matching the "throughout this paper" statement.

Throughout, we write $G = (V, E, \sigma)$ for a multigraph $G = (V, E)$ together with an edge-weight function $\sigma \in \mathbb{R}^E_{>0}$. However, σ is not part of the multigraph structure itself. We use the triple notation for modulus results, which is where the edge weights are relevant. . For the weight-agnostic combinatorial defintions and theorems (families of objects, densities, admissibility, the Fulkerson dual) we write the plain pair $G = (V, E)$ instead, since none of those notions depend on σ.

Definition 2 (Spanning tree)**.** A set of edges $T \subseteq E$ is a *spanning tree* of G if it touches every vertex of V and contains no cycle. In particular, this immediately rules out parallel edges, which would form a 2-cycle.

This is formalized in `Multigraph.IsSpanningTree`: T is acyclic (no loops, no parallel edges within T, and the simple graph obtained by forgetting edge identities is acyclic) and that same simple graph is connected, i.e. touches and joins every vertex of V.

1.2 Families of objects and Fulkerson duality

This section follows Section 1.5 of Albin, Clemens, Hoare, Poggi-Corradini, Sit, and Tymochko (2021). Families of objects generalize spanning trees (and paths, cuts, etc.) to a common framework, in which the notion of an admissible density and its Fulkerson dual family are defined. Spanning trees are recovered as the special case worked out in Section 2.2.

The machinery of Fulkerson duality has been developed in the context of finite families of objects (such as the spanning trees of a finite graph). The results in this section are stated for finite families, but the definitions are given in a more general context that allows for countably infinite families as well.

Definition 3 (Family of objects)**.** Let $G = (V, E)$ be a multigraph. A *family of objects* on G is a nonempty countable set $\Gamma \subseteq \mathbb{R}^E_{\geq 0}$ (η-space). In practice, each each $\gamma \in \Gamma$ is associated with graph-theoretic object (a spanning tree, path, cut, etc.) identified with its *usage vector*: $\gamma(e)$ is the usage of edge e by γ.

Definition 4 (Family with no zero object)**.** A family of objects Γ *has no zero object* if every $\gamma \in \Gamma$ has $\gamma(e) > 0$ for some edge $e \in E$, i.e. $\gamma \neq 0$. This rules out the trivial all-zero object, for which no density could ever be admissible.

Definition 5 (Density)**.** A *density* is a point $\rho \in \mathbb{R}^E_{\geq 0}$ that assigns a nonnegative cost $\rho(e)$ to each edge $e \in E$.

Definition 6 (Length of an object)**.** Let ρ be a density on G. The *length* of an object $\gamma \in \Gamma$ with respect to ρ is

$$\ell_\rho(\gamma) := \sum_{e \in E} \gamma(e)\rho(e).$$

Implementation note: the sum is implemented with `finsum` *rather than* `Finset.sum` *over* `Finset.univ`*, so that it only requires* `Finite E`*, not a* `Fintype` *instance.* See `docs/common.md` for details.

Definition 7 (Admissible density)**.** A density $\rho \in \mathbb{R}^E_{\geq 0}$ is *admissible* for a family Γ if $\ell_\rho(\gamma) \geq 1$ for every $\gamma \in \Gamma$. The *admissible set* of Γ is defined as

$$\mathrm{Adm}(\Gamma) := \{\rho \in \mathbb{R}^E_{\geq 0} : \ell_\rho(\gamma) \geq 1 \ \forall \gamma \in \Gamma\}.$$

Definition 8 (Fulkerson dual family)**.** Let Γ be a finite family of objects on $G = (V, E)$. The *Fulkerson dual family* of Γ is $\hat{\Gamma} := \mathrm{ext}(\mathrm{Adm}(\Gamma)) \subset \mathbb{R}^E_{\geq 0}$, the set of extreme points of $\mathrm{Adm}(\Gamma)$.

Theorem 9 (Fulkerson duality)**.** *For any finite family of objects* Γ, $\hat{\hat{\Gamma}} \subseteq \Gamma$.

1.3 Modulus of families of objects

This section follows Section 1.6 of Albin, Clemens, Hoare, Poggi-Corradini, Sit, and Tymochko (2021).

Definition 10 (Energy of a density)**.** Let $1 \leq p \leq \infty$ and let $G = (V, E, \sigma)$ be a multigraph. The *energy* of a density ρ is

$$\mathcal{E}_{p,\sigma}(\rho) := \begin{cases} \sum_{e \in E} \sigma(e)\rho(e)^p & 1 \leq p < \infty, \\ \max_{e \in E} \sigma(e)\rho(e) & p = \infty. \end{cases}$$

Definition 11 (p-modulus)**.** Let $G = (V, E, \sigma)$ be a multigraph. Fix $1 \leq p \leq \infty$. The *p-modulus* of a family of objects Γ on G is

$$\mathrm{Mod}_{p,\sigma}(\Gamma) := \inf_{\rho \in \mathrm{Adm}(\Gamma)} \mathcal{E}_{p,\sigma}(\rho).$$

A density achieving this infimum is called *extremal* (or optimal).

Definition 12 (Extremal density)**.** A density $\rho^* \in \mathrm{Adm}(\Gamma)$ is *extremal* (or optimal) for $\mathrm{Mod}_{p,\sigma}(\Gamma)$ if

$$\mathrm{Mod}_{p,\sigma}(\Gamma) = \mathcal{E}_{p,\sigma}(\rho^*).$$

Theorem 13 (Existence and uniqueness of the extremal density)**.** *Let Γ be a finite family of objects on $G = (V, E, \sigma)$ with $\mathrm{Adm}(\Gamma) \neq \emptyset$, and fix $1 < p < \infty$. Then there is a unique density $\rho^* \in \mathrm{Adm}(\Gamma)$ achieving $\mathrm{Mod}_{p,\sigma}(\Gamma)$.*

Theorem 14 (Fulkerson duality for modulus)**.** *Let $G = (V, E, \sigma)$ be a multigraph. Let Γ be a finite family of objects on G. Let $1 < p < \infty$ and let q satisfy $pq = p + q$. Then*

$$\mathrm{Mod}_{p,\sigma}(\Gamma)^{1/p} \, \mathrm{Mod}_{q,\sigma^{1-q}}(\hat{\Gamma})^{1/q} = 1,$$

and the extremal density η^* for $\mathrm{Mod}_{q,\sigma^{1-q}}(\hat{\Gamma})$ is related to the extremal density ρ^* for $\mathrm{Mod}_{p,\sigma}(\Gamma)$ by

$$\eta^*(e) = \frac{\sigma(e)\rho^*(e)^{p-1}}{\mathrm{Mod}_{p,\sigma}(\Gamma)} \qquad \forall e \in E.$$

In particular, when $p = q = 2$,

$$\mathrm{Mod}_{2,\sigma}(\Gamma) \, \mathrm{Mod}_{2,\sigma^{-1}}(\hat{\Gamma}) = 1, \qquad \eta^*(e) = \frac{\sigma(e)}{\mathrm{Mod}_{2,\sigma}(\Gamma)} \rho^*(e).$$

Chapter 2

Fairest Edge Usage and Minimum Expected Overlap for Random Spanning Trees

This chapter formalizes results from Albin, Clemens, Hoare, Poggi-Corradini, Sit, and Tymochko (2021). It specializes the general modulus theory of Chapter 1 to spanning trees, and develops the paper's two main optimization problems: minimum expected overlap (MEO) and fairest edge usage (FEU).

2.1 Random spanning trees, MEO, and FEU

This section follows Sections 1.1–1.3 of the paper: it sets up random spanning trees and states the two optimization problems, minimum expected overlap (MEO) and fairest edge usage (FEU), whose relationship to spanning tree modulus is the subject of the rest of the paper.

Definition 15 (Law of a random spanning tree)**.** Let $G = (V, E)$ be a finite, connected multigraph and let Γ_G be its (finite) set of spanning trees. A *law* on Γ_G is a probability mass function $\mu\colon \Gamma_G \to [0, 1]$ with $\sum_{\gamma\in\Gamma_G} \mu(\gamma) = 1$; we write $\mu \in P(\Gamma_G)$. A random spanning tree $\underline{\gamma}$ *has law* μ (written $\underline{\gamma} \sim \mu$) if $\mathbb{P}_\mu(\underline{\gamma} = \gamma) = \mu(\gamma)$ for every $\gamma \in \Gamma_G$.

Definition 16 (Minimum expected overlap problem)**.** For $\mu \in P(\Gamma_G)$, let $\underline{\gamma}, \underline{\gamma}' \sim \mu$ be independent. The *expected overlap* of μ is

$$\mathbb{E}_\mu|\underline{\gamma} \cap \underline{\gamma}'| := \sum_{\gamma,\gamma'\in\Gamma_G} |\gamma \cap \gamma'|\, \mu(\gamma)\mu(\gamma').$$

The *minimum expected overlap (MEO) problem* is

$$\mathrm{MEO}(\Gamma_G) := \min_{\mu\in P(\Gamma_G)} \mathbb{E}_\mu|\underline{\gamma} \cap \underline{\gamma}'|.$$

A law achieving this minimum is *optimal* for $\mathrm{MEO}(\Gamma_G)$.

Definition 17 (Edge usage probability)**.** For $\mu \in P(\Gamma_G)$ and $e \in E$, the *edge usage probability* of e under μ is

$$\mathbb{P}_\mu(e \in \underline{\gamma}) := \sum_{\gamma \in \Gamma_G} \mathbf{1}_{\{e \in \gamma\}}\, \mu(\gamma).$$

Definition 18 (Fairest edge usage problem)**.** The *fairest edge usage (FEU) problem* asks for the edge usage probability function $\eta \colon E \to [0,1]$ of minimal variance over $\mu \in P(\Gamma_G)$:

$$\text{minimize} \ \ \mathrm{Var}(\eta) \quad \text{subject to} \quad \eta(e) = \mathbb{P}_\mu(e \in \underline{\gamma}) \ \forall e \in E, \quad \mu \in P(\Gamma_G).$$

We write $\mathrm{FEU}(\Gamma_G)$ for this problem.

Definition 19 (Fair and forbidden trees)**.** A spanning tree $\gamma \in \Gamma_G$ is a *fair tree* if $\mu^*(\gamma) > 0$ for some law μ^* optimal for $\mathrm{FEU}(\Gamma_G)$. The set of fair trees is denoted Γ_G^f. A tree $\gamma \in \Gamma_G \setminus \Gamma_G^f$ (if any exist) is a *forbidden tree.*

2.2 Spanning tree modulus and the MEO problem

This section follows Section 1.7 of the paper and connects the general modulus framework of Chapter 1 back to spanning trees, recovering MEO and FEU as modulus problems. From here on the paper restricts to $p = 2$ and unweighted graphs ($\sigma \equiv 1$), and we do the same.

2.2.1 Strong duality for 2-modulus

The general existence/uniqueness and Fulkerson duality theorems of Section 1 (Theorems 13 and 14) are stated for general $1 < p < \infty$ but are not yet formalized in that generality. The results below are the $p = 2$ specializations actually used by this chapter; they are proved directly using the connection between the two-norm and the inner product.

Theorem 20 (Existence and uniqueness of the 2-extremal density)**.** *Let Γ be a finite family of objects on $G = (V, E, \sigma)$ with $\mathrm{Adm}(\Gamma) \neq \emptyset$. There is a unique density $\rho^* \in \mathrm{Adm}(\Gamma)$ achieving $\mathrm{Mod}_{2,\sigma}(\Gamma)$.*

Theorem 21 (Strong duality for 2-modulus)**.** *Let Γ be a finite family of objects on $G = (V, E, \sigma)$ with $\mathrm{Adm}(\Gamma) \neq \emptyset$, and let ρ^* be the extremal density for $\mathrm{Mod}_{2,\sigma}(\Gamma)$. There exists $\lambda^* \in \mathbb{R}^\Gamma_{\geq 0}$ such that*

$$\rho^*(e) = \frac{1}{2\sigma(e)} \sum_{\gamma \in \Gamma} \gamma(e) \lambda^*(\gamma) \qquad \forall e \in E,$$

$$\lambda^*(\gamma)(1 - \ell_{\rho^*}(\gamma)) = 0 \qquad \forall \gamma \in \Gamma,$$

and

$$\mathrm{Mod}_{2,\sigma}(\Gamma) = \sum_{\gamma \in \Gamma} \lambda^*(\gamma) - \frac{1}{4} \sum_{e \in E} \frac{1}{\sigma(e)} \Big(\sum_{\gamma \in \Gamma} \gamma(e) \lambda^*(\gamma) \Big)^2.$$

Corollary 22 (Weak duality for the Lagrangian dual)**.** *For any $\rho \in \mathrm{Adm}(\Gamma)$ and any $\lambda \in \mathbb{R}^\Gamma_{\geq 0}$,*

$$\mathcal{E}_{2,\sigma}(\rho) \ \geq \ \sum_{\gamma \in \Gamma} \lambda(\gamma) - \frac{1}{4} \sum_{e \in E} \frac{1}{\sigma(e)} \Big(\sum_{\gamma \in \Gamma} \gamma(e) \lambda(\gamma) \Big)^2,$$

with equality if and only if ρ *and* λ *satisfy the stationarity and complementary slackness conditions of Theorem 21.*

Corollary 23 (2-modulus Fulkerson duality)**.** *In the setting of Theorem 21, define* $\eta^*(e) := \sigma(e)\rho^*(e)/\operatorname{Mod}_{2,\sigma}(\Gamma)$ *for* $e \in E$*. Then* $\eta^* \in \operatorname{Adm}(\hat{\Gamma})$ *and* η^* *is extremal for* $\operatorname{Mod}_{2,\sigma^{-1}}(\hat{\Gamma})$*, with*

$$\operatorname{Mod}_{2,\sigma}(\Gamma)\operatorname{Mod}_{2,\sigma^{-1}}(\hat{\Gamma}) = 1.$$

This recovers the $p = q = 2$ *case of Theorem 14 without the general machinery.*

Definition 24 (Feasible partition)**.** A *feasible partition* of $G = (V, E)$ is a partition $P = \{V_1, \dots, V_{k_P}\}$ of V into $k_P \geq 2$ parts such that each induced subgraph $G(V_i)$ is connected. Its edge set $E_P \subseteq E$ consists of the edges joining vertices belonging to different parts.

Theorem 25 (Fulkerson dual of spanning trees)**.** *Let* Γ_G *be the (finite) family of spanning trees of* $G = (V, E)$*, with usage given by the indicator vectors* $\mathbf{1}_{\{e\in\gamma\}}$*. The Fulkerson dual family* $\widehat{\Gamma_G}$ *is the set of vectors* $\frac{1}{k_P-1}\mathbf{1}_{E_P}$*, ranging over all feasible partitions* P *of* G*.*

Theorem 26 (Spanning tree modulus, MEO, and FEU)**.** *Let* $\Gamma = \Gamma_G$ *be the (finite) family of spanning trees of* $G = (V, E)$ *and let* $\hat{\Gamma}$ *be its Fulkerson dual. Densities* $\rho, \eta \in \mathbb{R}^E_{\geq 0}$ *and a law* $\mu \in P(\Gamma)$ *are simultaneously optimal for* $\operatorname{Mod}_2(\Gamma)$*,* $\operatorname{Mod}_2(\hat{\Gamma})$*, and* $\operatorname{MEO}(\Gamma)$ *respectively if and only if*

$$\rho \in \operatorname{Adm}(\Gamma), \quad \eta(e) = \sum_{\gamma\in\Gamma} \gamma(e)\,\mu(\gamma)\ \forall e \in E, \quad \eta(e) = \frac{\rho(e)}{\operatorname{Mod}_2(\Gamma)}\ \forall e \in E, \quad \mu(\gamma)(1-\ell_\rho(\gamma)) = 0\ \forall \gamma \in \Gamma.$$

In particular,

$$\operatorname{MEO}(\Gamma) = \operatorname{Mod}_2(\hat{\Gamma}) = \operatorname{Mod}_2(\Gamma)^{-1}.$$

Corollary 27 (Weak duality for modulus)**.** *Let* Γ *be a finite family of objects on* G *with Fulkerson dual* $\hat{\Gamma}$*. For any* $\rho \in \operatorname{Adm}(\Gamma)$ *and* $\eta \in \operatorname{Adm}(\hat{\Gamma})$*,*

$$E_2(\rho)E_2(\eta) \geq 1, \qquad \text{where } E_2(\rho) := \sum_{e\in E} \sigma(e)\rho(e)^2,$$

with equality if and only if ρ *and* η *are optimal for* $\operatorname{Mod}_2(\Gamma)$ *and* $\operatorname{Mod}_2(\hat{\Gamma})$ *respectively.*